A Quantum Approach To Static Games Of Complete Information *

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Abstract

We extend the concept of a classical two-person static game to the quantum domain, by giving an Hilbert structure to the space of classical strategies and studying the *Battle of the Sexes* game. We show that the introduction of entangled strategies leads to a unique solution of this game.

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1 Introduction.

Classical Game Theory concerns the study of multiperson decision problems, where two or more individuals make rational decisions that will influence one another's welfare. The kind of problems the theory faces off arise frequently in economics and in social sciences, ranging from the conflicts of firms in the market to the foreign policy of nations. As far as physics is concerned, there is an intimate connection between the theory of games and quantum communication theory, where the task of two distant players is to obtain as much information as possible in a given physical situation.

Modern Game Theory was first developed in 1944 in the seminal book of Von Neumann and Morgenstern [1], but a better comprehension of its formal structure and the major mathematical results were achieved in the following years at Princeton, mainly due to the contribution of the young brilliant mathematician John Nash. The theory tries to understand the birth and the development of conflictual or cooperational behaviours between rational and intelligent decision makers, by analyzing simple but paradigmatic problems, which retain the crucial features exhibited by the real situations in every day life. It is worthwhile to stress that in the theoretical

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language a game refers to any situation where two or more persons, called players, are involved in a cooperation or a competition among themselves to reach a final state recognized as the best goal they can obtain from a cooperative or an individual point of view. It is clear that this general definition encompasses the usual meaning given to a game, i.e. that of a play (like chess and draughts), but it applies to bargaining or trading processes too. The persons who play the game are assumed to think, and consequently act, in a rational way, by trying to maximize the expected values of some functions, the payoff functions, which depend on the choices of all the players and represent their gain at the end of the game. They are also considered as intelligent beings, in the sense that each player knows about the game everything that we know: therefore they can make the same inferences about the development of the game that we are able to do.

The purpose of this paper is to try to formalize the concept of Quantum Game (as opposed to Classical Game), by giving a formal quantum structure to the usual mathematical tools commonly used in analysing decision problems. We have taken inspiration from the paper by J. Eisert, M. Wilkens and M. Lewenstein [2], but the formal structure of the Quantum Game Theory we are trying to develop differs completely from their approach, though holding true that we agree about the special role played by entanglement.

The passage from a classical to a quantum domain will be achieved by using an Hilbert space of strategies, instead of a discrete set of them, then allowing the possibility of the existence of linear superpositions between classical strategies: this will naturally enlarge the possible strategic choices offered to each player from a numerable to a continuous set.

As it will be evident from the analysis of one of the famous and paradigmatic classical games (the Battle of the Sexes), the use of a quantum formalism will exhibit a novel feature, the appearance in *entangled strategies* of a Nash Equilibrium representing the unique solution of the game – in contrast with what happens in the classical case, where the theory is not able to make any unique prediction.

In sections 2 and 3 the basic concepts of Classical Game Theory are enunciated and applied to the study of the forementioned game. In section 4 we try to define what is meant by a quantum strategy, as opposed to a classical one, and in sections 5 and 6 we show how it could be exploited by each player to find out the Nash equilibria in the case of factorizable Quantum Strategies. In section 7 we extend the method to the entangled strategies, showing that a unique solution arises for the Battle of the Sexes game.

2 Classical Game Theory

In this chapter we want to provide an introductive view of the basic principles of the theory of the Static Games of Complete Information. Simple decision problems, in which both players simultaneously choose their actions (Static Game) and receive a payoff depending on their mutual choices, are contained within this restricted class of games. Moreover, each player knows perfectly the values of the payoff functions of all his opponents (Complete Information). Games which do not belong to this class are Dynamic Games (the players play sequential moves by turn, like in chess) or Games of Incomplete Information (some player does not know exactly the payoff functions of his opponents, like in a auction).

Every **Normal Form Representation** of our games must specify:

1. the number of players $i = 1 \dots N$;
2. the discrete set of strategies $\{s\}$, which constitutes the *strategic space* $\mathcal{S}_i$ available to each player;
3. the payoff functions $\$_i = \$_i(s_1, s_2 \dots s_N)$, which assign the i-th player a real number depending on the strategies chosen by his opponents.

These functions quantify the benefit that the i-th player gains if the strategies $(s_1, s_2 \dots s_N)$ are effectively played, and the purpose of each decision maker consists in trying to maximize, in a sense depending on the context of the game, this real number.

The theory of games deals therefore with the forecast of which will be the most rational development of a given game: as we will see later, it will not necessarily be the most appealing one for each player, but the one that each player is forced to play in order to avoid major losses.

In order to simplify the notation, we will denote from now on by s_i the generic strategy belonging to the strategic space $\mathcal{S}_i$ of the i-th player. Each player chooses his own strategy simultaneously to the others, (i.e. without knowing what the other players are going to do, but knowing what they might do) and has a complete information about the consequences of all the possible choices.

Having settled the common framework of all the Static Games, it is useful to introduce the notion of a strictly dominated strategy. Strategy s_i is **strictly dominated** by strategy $\tilde{s_i}$, if:

$$\$_i(s_1, \dots, s_i, \dots, s_N) < \$_i(s_1, \dots, \tilde{s_i}, \dots, s_N), \tag{2.1}$$

for every set of strategies $(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_N)$ of the other players.

This concept suggests a first kind of solution for a given game, where the word solution refers to the ensemble of the N strategies which are actually played. In fact, making the reasonable assumption that rational players do not play strictly dominated strategies (as we have said before, they pursue the goal of maximizing their benefits, whatever are the strategies of their opponents), we could find out the best strategy for each player by simply eliminating all the dominated ones. The strategy obtained by such a procedure, which is termed *Iterated Elimination of Strictly Dominated Strategies*, represents the best action each player could reasonable perform against his opponent's decisions. Therefore the set of strategies chosen in such a way constitutes a possible (rational) solution of the game.

Unfortunately, this appealing procedure produces no prediction at all for certain classes of problems, where no strategy survives the process of elimination. In such a case it is not clear which action could be considered a rational and optimal decision.

In order to illustrate the concepts we have just introduced, we analyse a simple two-person static game of complete information, which is known as the Battle of the Sexes. In the usual exposition of the game a woman, Alice, and a man, Bob, are trying to decide where to spend the saturday evening: Alice would be happy to attend the Opera, while Bob would like to watch the football match at the Television, and both of them would be happier to stay together rather than far apart. We can represent this simple game in a normal form by denoting by O (Opera)

and T (Television) the two strategies which constitute the common strategic space $\mathcal{S}$ and we can represent their payoff functions by means of a bimatrix like the following one:

		Bob	
		O	**T**
Alice	**O**	(α, β)	(γ, γ)
	T	(γ, γ)	(β, α)

where α, β, γ are the values assumed by the payoff functions of Alice and Bob in correspondence to the possible choice of strategies: if, for example, both players decide to go to the Opera (i.e., if the couple of strategies (O, O) is played), Alice gains $\$_A(O, O) = \alpha$ and Bob gains $\$_B(O, O) = \beta$. In order to satisfy the preferences of the two players mentioned before, the condition $\alpha > \beta > \gamma$ must be imposed. This constraint condition guarantees that we are dealing with a proper form of the Battle of the Sexes game, with the right options for Alice and Bob.

This is a peculiar game, where strictly dominated strategies do not exist (in fact according to Alice, for example, strategy O is better than T if Bob decides to play O, and it is worse if Bob decides to play T): neither of the players can rationally eliminate one of his strategies and prefer the other.

To try to overcome this unpleasant situation, which is not peculiar of the Battle of the Sexes game, one is induced to resort in general to the concept of Nash Equilibrium, which permits it to get possible and rational solutions for the kind of games we are referring to (it is worthwhile remembering that the word *solution* corresponds to a set of strategies the rational players will surely play). In a N-player normal form representation game, the set of strategies $(s_1^\star, s_2^\star, \ldots, s_N^\star)$ constitute a **Nash Equilibrium** if, for each player i, the strategy $s_i^\star$ satisfies the following equation:

$$\$_i(s_1^\star, \ldots, s_{i-1}^\star, s_i^\star, s_{i+1}^\star, \ldots, s_N^\star) \geq \$_i(s_1^\star, \ldots, s_{i-1}^\star, s_i, s_{i+1}^\star, \ldots, s_N^\star), \tag{2.2}$$

for every strategy s_i belonging to the i-th strategic space $\mathcal{S}_i$.

Therefore a Nash Equilibrium corresponds to a set of strategies, each one representing the best choice for each single player if all his opponents take their best decision too. It exhibits the appealing property of being *strategically stable*, because no single player has the incentive to deviate unilaterally from his strategy without suffering some loss in his own payoff. The motivation leading to this new concept is based on the following properties:

1. it generalizes the previous and inadequate concept of *solution of a game*, for it can be easily proved that every set of strategies surviving the process of elimination of strictly dominated strategies constitutes the unique Nash Equilibrium of the game – the converse statement being not necessarily true. We will in particular see that the Battle of the Sexes

possesses two Nash Equilibria in pure strategies (representing two possible predictions for the development of the game), while the process of elimination of strategies cannot give any result;

2. it is, as already said, a *stable* equilibrium, in the sense that no player could gain better by unilaterally deviating from his predicted strategy.

Returning back to the analysis of our game and in order to find out which is the pair of strategies which satisfy the Nash-Equilibrium conditions (2.2), we must solve the two following coupled inequalities in the unknown strategies $s_A^\star$ and $s_B^\star$, where the letters A and B stand obviously for Alice and Bob:

$$\begin{cases} \$_A(s_A^\star, s_B^\star) \geq \$_A(s_A, s_B^\star) \qquad \forall\ s_A \in \mathcal{S}, \\ \\ \$_B(s_A^\star, s_B^\star) \geq \$_B(s_A^\star, s_B) \qquad \forall\ s_B \in \mathcal{S}. \end{cases} \tag{2.3}$$

It is easy to verify that two couples of strategies satisfy these conditions for the Battle of the Sexes Game: there exist therefore two Nash Equilibria, corresponding to the choices $(s_A^\star = O\,,\ s_B^\star = O\,)$ or $(s_A^\star = T\,,\ s_B^\star = T\,)$. These strategies are strategically stable exactly in the sense given above.

The appearance of two possible solutions of the game in this peculiar case is rather discomforting (being the first one more favourable to Alice, while the second is favourable to Bob), even if this is better than having no solution, as it was the case before the Nash Equilibrium was introduced. However, as we will show in the next sections, this problem disappears in a quantum context, where we will be able to exhibit (in peculiar situations to be specified) only one solution of the game.

3 Mixed Classical Strategies

Although the notion of a Nash Equilibrium represents a very clever new concept of solution for a game, it is still possible to exhibit simple games not possessing Nash Equilibria, this fact leading to a situation that does not display any rational prediction about the development of the game. This usually happens in games where each player would like to outguess the decisions taken by his opponents (like in poker or in a battle) in order to gain an advantage over them. To solve this new kind of games we must resort to the introduction of the concept of the **Mixed Classical Strategies**, as opposed to the Pure ones presented before: in a normal form representation game, a Mixed Strategy for the i-th player is a probability distribution which assigns a probability p_i^α to each pure strategy s_i^α, where α runs over all the possible strategies belonging to the strategic space $\mathcal{S}_i$ of the i-th player.

This means that each player chooses one of the possible strategies by resorting to chance, according to the probability distribution which could assure him the best result.

This definition generalizes the notion of strategy given before, since a pure strategy can always be seen as a mixed one with probability distribution which is equal to one for a particular strategy and zero for all the others. According to Harsanyi [3], it is possible to interpret such a mixed strategy in terms of the uncertainty of each player about what another player will do.

The introduction of a probability distribution in the space of strategies forces one to modify the concept of the payoff function for the i-th player, leading to an **expected payoff function** $\bar{\$}_i$:

$$\bar{\$}_i(\{p_1\}, \{p_2\}, \ldots, \{p_N\}) = \sum_{\alpha,\beta,\ldots,\gamma} p_1^\alpha p_2^\beta \ldots p_N^\gamma \, \$_i(s_1^\alpha, s_2^\beta, \ldots, s_N^\gamma) \tag{3.1}$$

where $\{p_i\}$ is the set of the probabilities associated to the strategies of the space $\mathcal{S}_i$.

We note that $\bar{\$}_i$ is no longer a function of the strategies, depending now on their probability distributions and represents an average gain for the i-th player.

The usefulness in dealing with mixed strategies resides in a remarkable result proved by Nash [4], stating that in every N-player normal form game with finite strategic spaces there always exists at least one Nash Equilibrium, in pure or mixed strategies. Therefore, one is led to modify slightly the definition of Nash Equilibrium in order to incorporate mixed strategies too. In the two-player normal form games with two strategies, like the Battle of the Sexes, we are concerned with the probabilities $\{p_1^1, p_1^2 = 1 - p_1^1\}$ for the first player, and $\{p_2^1, p_2^2 = 1 - p_2^1\}$ for the second one.

Since only two of them are independent, we put $p_1^1 = p$ and $p_2^1 = q$, writing $\bar{\$}_i(\{p_1\}, \{p_2\}) \equiv \bar{\$}_i(p, q)$. Using this simplified notation, we say that the mixed strategies characterized by the probabilities $(p^\star, q^\star)$ correspond to a Nash Equilibrium if the expected payoff functions satisfy the following conditions:

$$\begin{cases} \bar{\$}_1(p^\star, q^\star) \geq \bar{\$}_1(p, q^\star) & \forall \ p \in \ [0,1], \\ \\ \bar{\$}_2(p^\star, q^\star) \geq \bar{\$}_2(p^\star, q) & \forall \ q \in \ [0,1]. \end{cases} \tag{3.2}$$

We are now equipped with a more powerful mathematical tool, which can be applied to the analysis of the Battle of the Sexes. Let us suppose that Alice decides to resort to a probabilistic strategy in which she chooses Opera (s_1^1) with probability p (and obviously Television (s_1^2) with probabilty $1-p$) and Bob does the same, but with probability q for Opera (s_2^1) and $1-q$ for Television (s_2^2).

We can then calculate their expected payoff functions through equation (3.1), i.e. by summing all the joined probabilities relative to each couple of strategies, multiplied by the proper coefficients which appear in the bimatrix of the game, obtaining:

$$\begin{aligned} \bar{\$}_A(p, q) &= p[q(\alpha - 2\gamma + \beta) + \gamma - \beta] + \beta + q(\gamma - \beta), \\ \bar{\$}_B(p, q) &= q[p(\alpha - 2\gamma + \beta) + \gamma - \alpha] + \alpha + p(\gamma - \alpha). \end{aligned} \tag{3.3}$$

Nash Equilibria can be found by imposing the two conditions:

$$\begin{cases} \bar{\$}_A(p^\star, q^\star) - \bar{\$}_A(p, q^\star) = (p^\star - p)[\, q^\star(\alpha + \beta - 2\gamma) - \beta + \gamma\,] \geq 0 & \forall \ p \in \ [0,1], \\ \\ \bar{\$}_B(p^\star, q^\star) - \bar{\$}_B(p^\star, q) = (q^\star - q)[\, p^\star(\alpha + \beta - 2\gamma) - \alpha + \gamma\,] \geq 0 & \forall \ q \in \ [0,1]. \end{cases} \tag{3.4}$$

They are satisfied if the two factors in each inequality have both the same sign. There are three possibilities:

1. $p^\star_{(1)} = 1, q^\star_{(1)} = 1$.
   In such a case both Alice and Bob choose to play O, i.e. a pure strategy. Their corresponding expected payoff functions are $\bar{\$}_A(1,1) = \alpha$ and $\bar{\$}_B(1,1) = \beta$, and since $\alpha > \beta > \gamma$ the following inequalities hold:

$$\begin{cases} \bar{\$}_A(1,1) - \bar{\$}_A(p,1) = (1-p)(\alpha-\gamma) \geq 0 \quad \forall\ p \in\ [0,1], \\ \\ \bar{\$}_B(1,1) - \bar{\$}_B(1,q) = (1-q)(\beta-\gamma) \geq 0 \quad \forall\ q \in\ [0,1]. \end{cases} \tag{3.5}$$

2. $p^\star_{(2)} = 0, q^\star_{(2)} = 0$.
   The strategy is again a pure one, both Alice and Bob play T, and $\bar{\$}_A(0,0) = \beta$ and $\bar{\$}_B(0,0) = \alpha$ (note that $\bar{\$}_A$ and $\bar{\$}_B$ are reversed with respect to the previous case). Moreover:

$$\begin{cases} \bar{\$}_A(0,0) - \bar{\$}_A(p,0) = p(\beta-\gamma) \geq 0 \quad \forall\ p \in\ [0,1], \\ \\ \bar{\$}_B(0,0) - \bar{\$}_B(0,q) = q(\alpha-\gamma) \geq 0 \quad \forall\ q \in\ [0,1]. \end{cases} \tag{3.6}$$

   These two Nash Equilibria correspond to the solutions already found using pure strategies only, but the introduction of mixed strategies allows for the existence of a third kind of solutions.

3. Inequalities (3.4) can be satisified also for $p^\star$ and $q^\star$ different from 0 or 1. In such a case the factors $p^\star - p$ and $q^\star - q$ can be positive or negative, depending on the values of p and q. Then, the only way in order to fulfil conditions (3.4) consists in equating to zero the coefficients of $p^\star - p$ and $q^\star - q$, obtaining:

$$p^\star_{(3)} = \frac{\alpha-\gamma}{\alpha+\beta-2\gamma}, \qquad q^\star_{(3)} = \frac{\beta-\gamma}{\alpha+\beta-2\gamma}\,, \tag{3.7}$$

   where $p^\star$ and $q^\star$ are correctly larger than zero and less than one. Moreover they correspond to a Nash Equilibrium for which the gains of both players coincide:

$$\bar{\$}_A(p^\star_{(3)}, q^\star_{(3)}) = \bar{\$}_B(p^\star_{(3)}, q^\star_{(3)}) = \frac{\alpha\beta-\gamma^2}{\alpha+\beta-2\gamma}. \tag{3.8}$$

   It can be easily shown that the expected payoff functions for both players give now strictly less reward than the other two feasible strategies (O,O) or (T,T), since:

$$\gamma < \bar{\$}_{A(B)}(p^\star_{(3)}, q^\star_{(3)}) < \beta < \alpha. \tag{3.9}$$

We note that the solution of the game corresponding to the third case gives a good example of the exact meaning of Nash Equilibrium. In fact, this peculiar concept of equilibrium can be defined only if all players are assumed to accept the same logic, i.e. that of achieving the best result if all the players play together at their best. In the last case analysed the equilibrium corresponds to the conditions $\bar{\$}_A(p^\star_{(3)}, q^\star_{(3)}) = \bar{\$}_A(p, q^\star_{(3)})$ and $\bar{\$}_B(p^\star_{(3)}, q^\star_{(3)}) = \bar{\$}_A(p^\star_{(3)}, q)$. This

means that if, for example, Alice accepts the logic of Nash Equilibrium but Bob does not, the last one could choose the value of q minimizing $\bar{\$}_A(p^\star_{(3)}, q)$ without suffering any loss of gain.

The introduction of mixed strategies has not solved the problem of Alice and Bob: even in this case the theory fails to say which one of the obtained Nash Equilibria represents the real development of the game, because they all satisfy our notion of rational solution and are therefore equally compelling.

We might now ask if it would be possible to obtain a unique solution for the problem of the Battle of the Sexes by modifying again the theoretical structure of the game, in resemblance to what one does when enriching the concept of pure strategy by defining a probability distribution over the strategic space. This task is feasible introducing the notion of **Quantum Strategy**, in connection with the concept of Entanglement.

4 Quantum Game Theory

The purpose of this section is that of extending the notion of a classical (pure or mixed) strategy by giving a richer structure to the strategic space, so that it contains the old set of strategies as a proper subspace. Therefore from now on, instead of considering only a discrete and finite set of strategies, we will permit the existence of linear superpositions of them, by giving the formal structure of an Hilbert space to the strategic space. This allows the possibility of obtaining the *Pure Quantum Strategies*, defined as linear combinations, with complex coefficients, of pure classical ones. According to the orthodox intepretation of quantum mechanics, the squared modula of these complex coefficients have to be interpreted as the probability of having played one particular pure classical strategy. It is worthwhile observing that this intepretation of pure quantum strategies does not differ from the classical notion of a mixed strategy introduced before, because we are presently dealing with a restricted class of games (static games), where typical quantum interference effects between amplitudes do not occur.

Let us now apply the quantum approach to the study of the Battle of the Sexes, assuming that the new space of strategies available to Alice and Bob consists of a two-dimensional Hilbert space, whose orthonormal basis vectors $|O\rangle$ and $|T\rangle$ correspond to classical strategies O and T, respectively.

An arbitrary pure quantum strategy is therefore described by the normalized state vector:

$$|\psi\rangle = a\,|O\rangle + b\,|T\rangle \ , \qquad\qquad aa^\star + bb^\star = 1. \tag{4.1}$$

In order to use the normal form representation, we must define the quantum analogues of classical expected payoff functions and to decide the way each player can pick out his *best* strategy, i.e. the strategy which fulfils the conditions to be a Nash Equilibrium, if it does exist. With the aim of solving these issues, we must make the following general assumptions, which we hold to represent the common theoretical ingredients of all quantum games we can devise.

1. Unlike in the classical game theory, we have to fix an arbitrary initial quantum state belonging to the Hilbert space $\mathcal{S} = \mathcal{S}_{\mathcal{A}} \otimes \mathcal{S}_{\mathcal{B}}$, obtained as a direct product of the two strategic spaces of the two players, whose orthonormal bases consist of the vectors associated to the pure classical strategies.

2. Each player can manipulate the initial state vector (his strategy) by performing some local transformation in order to obtain a *suitable* final state vector which will represent the quantum strategy the two players are going to play. In this approach, since the states correspond to the strategy, the operators related to the transformations correspond to the tactics to be employed in the game.

3. The expected payoff of each player must be evaluated by calculating first the squared modula of the projections of the final quantum state onto the basis vectors of space $\mathcal{S}$, and then by adding up the obtained numbers multiplied by the appropriate payoff coefficients (deducible from the payoff bimatrix).

4. Each player must eventually play the classical pure strategy, which results from a measurement process on the final quantum strategy, i.e. from a projection onto the canonical basis of his own strategic space.

Before proceeding on, it is worthwhile making some comments about these assumptions, in order to make clear their meaning and utility.

Having given an Hilbert structure to the strategic space of each player, it is natural to describe their mutual choices as state vectors belonging to the direct product of their Hilbert spaces, which includes a continuous number of possible strategies. Indeed, it is due to this richer structure that the Quantum Theory of Games exhibits more attractive features than its classical version: we will in fact be able to construct *factorizable strategies*, which are obtained as a direct product of two well-defined strategies for each single player, and *entangled strategies*, which cannot be decomposed in such a way and cannot be reduced to the previous ones by applying local transformations only. It will be seen that the presence of such an high degree of non-local correlations between the strategies of the two players will provide us with a unique Nash Equilibrium solution of the Battle of The Sexes.

As stated in the first of the previous rules, we have introduced the notion of the initial quantum strategy for the two players: this state has no classical analogue and no particular meaning in our quantum context, representing only a starting vector which the two players must manipulate in a suitable way in order to obtain the final strategy, this last one being the real important object.

If we start working with a factorizable initial state vector, the results (existence of Nash equilibria and value of expected payoff functions) of our treatment will not depend on its particular form, for each player can modify it with local operations in order to choose his own best strategy. Indeed, what is important in determining the reward given to each player, it is the form of the final state vector, which corresponds to the strategy effectively played by them[1]. This independence from the starting point will not hold true, as we will see, in the case of entangled strategies, forcing us to impose limitations on the form of the initial vector.

The second assumption deals with the possibility given to each player to properly manipulate the initial state vector in order to find out the best final strategy, i.e. the one which satisfies the condition of corresponding to a Nash Equilibrium. The local transformations involved will depend on a certain number of parameters appearing as coefficients of the final strategy and

[1]These considerations cannot be applied to reference [2], where the results appear to be strictly initial state dependent.

determining the value of expected payoff functions. These parameters are the quantum analogues of the p and q parameters encountered in the classical mixed strategies, and the goal of each player will consist in searching for their best values.

The choice of the operators to be used is constrained only by the demand that, when using factorizable quantum states, the quantum version of the theory of games should be able to reproduce exactly all the results of the classical theory, whenever the players use mixed classical strategies. We are therefore free to choose the operators, satisfying the forementioned constraint, in a such a way to make as simple as possible the calculations.

The third assumption deals with the probabilistic structure of the theory, and asserts that each player can calculate an expected payoff function, representing the average reward each player would receive if the game were repeated many times (it has exactly the same meaning as in classical theory), starting from the same final strategy.

Finally, the fourth assumption states that each player must effectively play the pure classical strategy which emerges after a process of measurement, to be performed on the state vector representing the compound final strategy. In fact, according to the usual postulates of Quantum Mechanics, after the measurement of a suitable observable, the final state vector collapses onto one of the vectors of the canonical basis of the strategic space (which are pure classical strategies, due to the first assumption).

5 Factorizable Quantum Strategies

Let us define the four-dimensional Hilbert space of common strategies $\mathcal{S} = \mathcal{S}_{\mathcal{A}} \otimes \mathcal{S}_{\mathcal{B}}$ of Alice and Bob for the Battle of the Sexes Game by giving its orthonormal basis vectors:

$$\mathcal{S} = \mathcal{S}_A \otimes \mathcal{S}_B = (|OO\rangle, |OT\rangle, |TO\rangle, |TT\rangle), \tag{5.1}$$

where the first position is reserved to the state of Alice and the second one to that of Bob. These vectors permit it to write all the possible pure classical strategies which Alice and Bob can simultaneously play, i.e. states of the type $[\, x|0\rangle + y|T\rangle \,] \otimes [\, w|O\rangle + z|T\rangle \,]$.

Let us call A and B two unitary and unimodular matrices, representing the transformations that Alice and Bob may respectively use to manipulate their own strategies (they are written in the canonical basis where the first vector is denoted by $|O\rangle$ and the second by $|T\rangle$):

$$A = \begin{bmatrix} a & b \\ -b^\star & a^\star \end{bmatrix}, \qquad B = \begin{bmatrix} c & d \\ -d^\star & c^\star \end{bmatrix}, \qquad \begin{cases} aa^\star + bb^\star = 1 \\ cc^\star + dd^\star = 1 \end{cases} . \tag{5.2}$$

Since a state of the type $r|O\rangle + s|T\rangle$ can always be obtained from states $|O\rangle$ or $|T\rangle$ through $SU(2)$ transformations, the present procedure will be invariant with respect to the choice of the initial state. Then, we can make the calculation particularly simple starting from one of the four states $|OO\rangle, |OT\rangle, |TO\rangle, |TT\rangle$. Let us then choose for example $|\psi_{in}\rangle = |OO\rangle$.

By applying operators (5.2) to the initial state, we obtain a final common quantum strategy depending on the four complex coefficients a, b, c, d:

$$|\psi_{fin}\rangle = A \otimes B\,|\psi_{in}\rangle = ac|OO\rangle - ad^\star|OT\rangle - b^\star c|TO\rangle + b^\star d^\star|TT\rangle. \tag{5.3}$$

It is now possible to compute the expected payoff functions of both players by projecting the final state onto the canonical basis of $\mathcal{S} = \mathcal{S}_{\mathcal{A}} \otimes \mathcal{S}_{\mathcal{B}}$ and applying the third quantum rule:

$$\begin{aligned} \bar{\$}_A &= |a|^2[(\alpha+\beta-2\gamma)|c|^2 - \beta + \gamma] + \beta + (\gamma-\beta)|c|^2, \\ \bar{\$}_B &= |c|^2[(\alpha+\beta-2\gamma)|a|^2 - \alpha + \gamma] + \alpha + (\gamma-\alpha)|a|^2, \end{aligned} \tag{5.4}$$

where we have eliminated the parameters b and d.

Now Alice and Bob can respectively find out which are the values of coefficients $|a|^2$ and $|c|^2$ satisfying the Nash Equilibrium condition, whose defining inequalities (3.2) remain valid also for Quantum Game Theory. By repeating calculations similar to those of section 3, it is easy to notice that the desired values for the coefficients are represented by the following pairs:

$$(|a|^2 = 0\,,\, |c|^2 = 0) \;\; or \;\; (|a|^2 = 1\,,\, |c|^2 = 1) \;\; or \;\; (|a|^2 = \frac{\alpha-\gamma}{\alpha+\beta-2\gamma}\,,\, |c|^2 = \frac{\beta-\gamma}{\alpha+\beta-2\gamma}) \tag{5.5}$$

The first and the second pair of values are the quantum analogues of the pure classical strategies in which both players choose the same strategy. The final quantum strategy and the relative expected payoff functions are easily obtained by substituting these values in equations (5.3) and (5.4):

$$\begin{aligned} (|a|^2 = 0\,,\, |c|^2 = 0) \quad &\Rightarrow \quad |\psi_{fin}\rangle = |TT\rangle \quad \Rightarrow \quad \bar{\$}_A = \beta \quad \bar{\$}_B = \alpha, \\ (|a|^2 = 1\,,\, |c|^2 = 1) \quad &\Rightarrow \quad |\psi_{fin}\rangle = |OO\rangle \quad \Rightarrow \quad \bar{\$}_A = \alpha \quad \bar{\$}_B = \beta. \end{aligned} \tag{5.6}$$

The third pair of values for coefficients $|a|^2$ and $|c|^2$ corresponds to a mixed classical strategy played with probabilities given by equation (3.7). In fact, the factorizable final quantum strategy[2] and the expected payoff functions for Alice and Bob assume the following form:

$$\begin{aligned} |\psi_{fin}\rangle &= \frac{1}{\alpha+\beta-2\gamma}\,[\,\sqrt{\alpha-\gamma}|O\rangle - \sqrt{\beta-\gamma}|T\rangle\,]\cdot[\,\sqrt{\beta-\gamma}|O\rangle - \sqrt{\alpha-\gamma}|T\rangle\,] \\ &\Rightarrow \quad \bar{\$}_A = \bar{\$}_B = \frac{\alpha\beta-\gamma^2}{\alpha+\beta-2\gamma}\,. \end{aligned} \tag{5.7}$$

The quantities $\bar{\$}_A$ and $\bar{\$}_B$ have the same values we found in the case of mixed classical strategies – see equation (3.8).

Due to the forementioned invariance, if we were started from another initial quantum state we would have obtained different values for coefficients $|a|^2$ and $|c|^2$ satisfying Nash Equilibrium conditions, but the corresponding expressions for the final quantum strategy and the expected payoff functions for both players would be unchanged (always apart from phase factors).

Concluding, we see that factorizable quantum strategies are able to reproduce exactly the same results obtained before, when dealing with the mixed classical version of the Battle of the Sexes game. The exploitation of a quantum formalism and, particularly, of factorizable strategies does not constitute an improvement with respect to the classical theory: the game remains undecidible, since there is no way to prefer one Nash strategy with respect to the other ones. Nevertheless, Quantum Game Theory contains the Classical one as a subset, and quantum factorizable strategies correspond to classical mixed ones.

[2] We have omitted for simplicity useless phase factors in the coefficients of the final states, since we are considering only static games, where there is no evolution.

6 Density Matrix Approach to Quantum Game Theory

Before beginning the study of the properties of entangled quantum strategies, we show how to rewrite the entire formalism developed so far using density matrices instead of state vectors, and introducing a new kind of transformations. It is worthwhile doing it in order to be able to make simple the mathematical calculations when entangled strategies are used, bearing in mind that, in the case of factorizable states, the solutions we got in the previous section must be obtained again.

Let us therefore define a unitary and hermitian operator C interchanging vectors $|O\rangle$ and $|T\rangle$:

$$\left\{ \begin{array}{l} C\,|O\rangle = |T\rangle \\ C\,|T\rangle = |O\rangle \end{array} \right. \quad , \quad C^{\dagger} = C = C^{-1}\,. \tag{6.1}$$

We assume now that each player can modify his own strategy by applying to his reduced part of the total density matrix ρ_{in}, which represents the initial state of the game, the following transformation:

$$\rho_{fin}^{A(B)} = [\, p\, I\, \rho_{in}^{A(B)}\, I^{\dagger} + (1-p)\, C\, \rho_{in}^{A(B)}\, C^{\dagger}\,]\,. \tag{6.2}$$

This operation can be interpreted as the choice of each player to act with the identity I with probability p and with C with probability $(1-p)$, and gives rise to the following final density matrix:

$$\begin{aligned} \rho_{fin} \;=\;& pq\, I_A \otimes I_B\, \rho_{in}\, I_A^{\dagger} \otimes I_B^{\dagger} + p(1-q) I_A \otimes C_B\, \rho_{in}\, I_A^{\dagger} \otimes C_B^{\dagger} + \\ & q(1-p) C_A \otimes I_B\, \rho_{in}\, C_A^{\dagger} \otimes I_B^{\dagger} + (1-p)(1-q) C_A \otimes C_B\, \rho_{in}\, C_A^{\dagger} \otimes C_B^{\dagger} \end{aligned} \tag{6.3}$$

It is possible to show that, if we start from a density matrix corresponding to one of the four states $|OO\rangle, |OT\rangle, |TO\rangle$ and $|TT\rangle$ which we could have been used indifferently in the case of factorizable quantum strategies, the particular kind of transformation employed in equation (6.2) reproduces the same results obtained in section 5.

Let us choose for example the initial density matrix corresponding to the state $|\psi_{in}\rangle = |OO\rangle$ used in the preceding section. Then from equation (6.3) we obtain:

$$\begin{aligned} \rho_{fin} \;=\;& pq|OO\rangle\langle OO| + p(1-q)|OT\rangle\langle OT| + \\ & (1-p)q|TO\rangle\langle TO| + (1-p)(1-q)|TT\rangle\langle TT|. \end{aligned} \tag{6.4}$$

In order to calculate the payoff functions, we must introduce two payoff operators:

$$\begin{aligned} P_A &= \alpha|OO\rangle\langle OO| + \gamma(|OT\rangle\langle OT| + |TO\rangle\langle TO|) + \beta|TT\rangle\langle TT|, \\ P_B &= \beta|OO\rangle\langle OO| + \gamma(|OT\rangle\langle OT| + |TO\rangle\langle TO|) + \alpha|TT\rangle\langle TT|. \end{aligned} \tag{6.5}$$

The payoff functions can then be obtained as mean values of these operators:

$$\bar{\$}_A = Tr(P_A\, \rho_{fin}) \quad \bar{\$}_B = Tr(P_B\, \rho_{fin}). \tag{6.6}$$

Three Nash Equilibria arise: those for $(p^\star = 1, q^\star = 1)$ and for $(p^\star = 0, q^\star = 0)$, corresponding to $\rho_{fin} = |OO\rangle\langle OO|$ and $\rho_{fin} = |TT\rangle\langle TT|$, respectively, and a third one associated to the final density matrix:

$$\begin{aligned}\rho_{fin} \quad = \quad & \frac{1}{\alpha+\beta-2\gamma}[\,(\alpha-\gamma)|O\rangle\langle O| + (\beta-\gamma)|T\rangle\langle T|\,] \otimes \\ & \frac{1}{\alpha+\beta-2\gamma}[\,(\beta-\gamma)|O\rangle\langle O| + (\alpha-\gamma)|T\rangle\langle T|\,] \end{aligned} \tag{6.7}$$

The payoff functions corresponding to these strategies are the same we have already obtained in the previous section – see equations (5.6) and (5.7)[3]

Concluding, we have shown that it is possible to get the same results one obtains in the classical version of our game, by allowing the two players to manipulate their own strategy with unitary and unimodular operators or, equivalently, through a particular transformation involving two hermitian operators (C and I), one interchanging states $|O\rangle$ and $|T\rangle$ and the other leaving them unvaried.

7 Entangled Quantum Strategies

In sections 5 and 6 we have seen, using two different but equivalent ways, that the exploitation of factorizable quantum strategies does not modify at all the results we can obtain by applying classical game theory to the analysis of the Battle of the Sexes game. On the other hand, it is just the richer structure we have given to the strategic space of both players that allows us to obtain new results. In fact, up to here we have not introduced entangled states in the quantum version of the game: as a matter of fact, we are going to show how an entangled couple of strategies played by Alice and Bob will lead to a unique solution of the game, always satisfying the Nash Equilibrium condition.

According to what as been shown in the previous section, we can restrict ourselves to deal with density matrices associated to strategies, rather than to state vectors, and to use the transformations of equation (6.3). The choice of the initial density matrix is relevant, in so far as the payoff functions corresponding to the Nash Equilibria turn out to be all initial state dependent.

Let us start assuming that Alice and Bob have at their disposal the following entangled state:

$$|\psi_{in}\rangle = a|OO\rangle + b|TT\rangle \qquad |a|^2 + |b|^2 = 1\,, \tag{7.1}$$

with the associated density matrix:

$$\rho_{in} = |a|^2|OO\rangle\langle OO| + ab^\star|OO\rangle\langle TT| + a^\star b|TT\rangle\langle OO| + |b|^2|TT\rangle\langle TT|. \tag{7.2}$$

[3] We note that starting with the state $|TT\rangle$ one obtains the same Nash Equilibria but now $(p^\star = 1, q^\star = 1)$ corresponds to $\rho_{fin} = |TT\rangle\langle TT|$ and $(p^\star = 0, q^\star = 0)$ to $\rho_{fin} = |OO\rangle\langle OO|$. If we choose $|OT\rangle$ or $|TO\rangle$ as starting states, the same final density matrices are found for $(p^\star = 0, q^\star = 1)$ and $(p^\star = 1, q^\star = 0)$ or $(p^\star = 1, q^\star = 0)$ and $(p^\star = 0, q^\star = 1)$, respectively. The third Nash Equilibrium corresponding to the ρ_{fin} given by (6.7) is always found.

By using equation (6.3) we obtain the expression for the final density matrix ρ_{fin}, which depends on parameters a, b, p and q, and, through eq. (6.7), also the expected payoff functions for both players:

$$\begin{aligned} \bar{\$}_A(p,q) &= p[\,q(\alpha+\beta-2\gamma)-\alpha|b|^2-\beta|a|^2+\gamma\,]+ \\ &\quad q[\,-\alpha|b|^2-\beta|a|^2+\gamma\,]+\alpha|b|^2+\beta|a|^2, \\ \bar{\$}_B(p,q) &= q[\,p(\alpha+\beta-2\gamma)-\beta|b|^2-\alpha|a|^2+\gamma\,]+ \\ &\quad p[\,-\beta|b|^2-\alpha|a|^2+\gamma\,]+\beta|b|^2+\alpha|a|^2. \end{aligned} \tag{7.3}$$

Then, three Nash Equilibria arise from the following two inequalities:

$$\begin{cases} \bar{\$}_A(p^\star,q^\star)-\bar{\$}_A(p,q^\star)=(p^\star-p)[\,q^\star(\alpha+\beta-2\gamma)-\alpha|b|^2-\beta|a|^2+\gamma\,]\geq 0, \\ \bar{\$}_B(p^\star,q^\star)-\bar{\$}_B(p^\star,q)=(q^\star-q)[\,p^\star(\alpha+\beta-2\gamma)-\beta|b|^2-\alpha|a|^2+\gamma\,]\geq 0. \end{cases} \tag{7.4}$$

Let us examine separately the three different Nash Equilibria:

1. $p^\star_{(1)}=q^\star_{(1)}=1$.
   In such a case, the initial and the final density matrix representing the common strategies played by the players turn out to be equal and the expected payoff functions are:

$$\bar{\$}_A(1,1) = \alpha|a|^2+\beta|b|^2 \tag{7.5}$$
$$\bar{\$}_B(1,1) = \beta|a|^2+\alpha|b|^2 \tag{7.6}$$

2. $p^\star_{(2)}=q^\star_{(2)}=0$.
   In correspondence to these values the final density matrix is obtained by reversing the initial strategies ($\rho_{fin}=C_AC_B\rho_{in}C_B^\dagger C_A^\dagger$), and the corresponding expected payoff functions turn out to be interchanged with respect to the previous case:

$$\bar{\$}_A(0,0) = \beta|a|^2+\alpha|b|^2, \tag{7.7}$$
$$\bar{\$}_B(0,0) = \alpha|a|^2+\beta|b|^2 \tag{7.8}$$

3. $p^\star_{(3)}=\frac{(\alpha-\gamma)|a|^2+(\beta-\gamma)|b|^2}{\alpha+\beta-2\gamma}$, $q^\star_{(3)}=\frac{(\alpha-\gamma)|b|^2+(\beta-\gamma)|a|^2}{\alpha+\beta-2\gamma}$.

   Due to the condition $\alpha>\beta>\gamma$ one sees immediately that $0<p^\star_{(3)}<1$ and $0<q^\star_{(3)}<1$. The expected payoff functions of both players turn out to be the same, and display the following dependance on the parameters of the game:

$$\bar{\$}_A(p^\star_{(3)},q^\star_{(3)})=\bar{\$}_B(p^\star_{(3)},q^\star_{(3)})=\frac{1}{\alpha+\beta-2\gamma}[\,\alpha\beta+(\alpha-\beta)^2|a|^2|b|^2-\gamma^2\,] \tag{7.9}$$

We have found again three different Nash Equilibria for the game, whose expected payoff functions are so related:

$$\bar{\$}_{A(B)}(p^\star_{(3)}, q^\star_{(3)}) < \bar{\$}_{A(B)}(p^\star_{(1)}, q^\star_{(1)}) \quad , \quad \bar{\$}_{A(B)}(p^\star_{(3)}, q^\star_{(3)}) < \bar{\$}_{A(B)}(p^\star_{(2)}, q^\star_{(2)}). \tag{7.10}$$

This simply means that both Alice and Bob would prefer to play strategies $p^\star_{(1)} = q^\star_{(1)} = 1$ or $p^\star_{(2)} = q^\star_{(2)} = 0$ rather than the third one, nonetheless being unable to decide which one of the two. In fact:

$$\begin{aligned} \bar{\$}_A(1,1) - \bar{\$}_A(0,0) &= (\alpha - \beta)(|a|^2 - |b|^2), \\ \bar{\$}_B(1,1) - \bar{\$}_B(0,0) &= (\alpha - \beta)(|b|^2 - |a|^2). \end{aligned} \tag{7.11}$$

Therefore, for $|a| > |b|$ Alice would prefer the first Nash Equilibrium, while Bob prefers the second one, and the contrary happens if $|a| < |b|$. However, there is a case in which both Alice and Bob can make the same choice: it corresponds to starting with an initial state having $|a| = |b|$, i.e. with:

$$|\psi_{in}\rangle = \frac{1}{\sqrt{2}} [\, |OO\rangle + |TT\rangle \,] , \tag{7.12}$$

$$\rho_{in} = \frac{1}{2} [\, |OO\rangle\langle OO| + |OO\rangle\langle TT| + |TT\rangle\langle OO| + |TT\rangle\langle TT| \,] .$$

It is now trivial to find out which are the Nash Equilibria for this peculiar initial state. We have:

$$\begin{aligned} (p^\star = 0, q^\star = 0) \quad &\Rightarrow \quad \$_A = \$_B = \frac{(\alpha + \beta)}{2} \\ (p^\star = 1, q^\star = 1) \quad &\Rightarrow \quad \$_A = \$_B = \frac{(\alpha + \beta)}{2} \\ (p^\star = \frac{1}{2}, q^\star = \frac{1}{2}) \quad &\Rightarrow \quad \$_A = \$_B = \frac{\alpha + \beta + 2\gamma}{4} \end{aligned} \tag{7.13}$$

The fact that both players have the same expected payoff functions constitutes the most attractive feature of having exploited entangled strategies. In fact, now it is possible to choose a unique Nash Equilibrium by discarding the ones which give the players the lesser reward: the remaining couple of values for p and q will then represent the unique solution of the game, which may satisfy both players, in contrast to what happens in the case of factorizable strategies, where there is no way to choose one of the three equivalent strategies. In fact, from eq.s (7.13) one sees immediately that:

$$\bar{\$}_{A(B)}(p^\star = 0, q^\star = 0) \;=\; \bar{\$}_{A(B)}(p^\star = 1, q^\star = 1) \;>\bar{\$}_{A(B)}(p^\star = 1/2, q^\star = 1/2). \tag{7.14}$$

The entangled quantum strategy which corresponds to both values $(p^\star = 0, q^\star = 0)$ and $(p^\star = 1, q^\star = 1)$ is the same and it is precisely the entangled state we started from:

$$|\psi_{fin}\rangle = \frac{1}{\sqrt{2}} [\, |OO\rangle + |TT\rangle \,] . \tag{7.15}$$

We have therefore obtained the result that the state (7.15) represents the quantum entangled pair of strategies which satisfy the Nash Equilibrium conditions, i.e. it represents the best rational couple of choices which are stable against unilateral deviations, and gives an higher reward to both players than the other possible Nash Equilibrium ($p^\star = 1/2, q^\star = 1/2$).

The entangled strategy (7.15) can therefore be termed the unique solution of the quantum version of the Battle of the Sexes game.

8 Conclusions

In this paper the classical theory of games has been extended to a quantum domain, by giving an Hilbert structure to the strategic spaces of the players, so allowing the existence of linear combinations of classical strategies to be interpreted according to the usual formalism of orthodox quantum mechanics. The only remarkable assumption we have made, reproducing the usual results of the classical game theory in the case of factorizable quantum strategies, is that each player must choose his strategy by performing quantum measurements on his state vector. Another weak assumption deals with the form of the operator which both players can apply in order to properly manipulate their strategies: we have shown that is sufficient, to reproduce the classical results and to obtain new ones, to restrict our attention to a transformation involving two particular hermitian operators.

We have applied this formalism to the study of the Battle of the Sexes game, in order to find out in which cases a quantum strategy could exhibit more attractive features than the classical theory predicts; it has been shown that the use of factorizable quantum strategies (i.e. factorizable states in the strategic space) by both players, cannot improve their expected payoff functions and reproduces the same behaviour of the classical version of the game: the Battle admits three Nash Equilibria which can be considered as three possible solutions of the game and the players cannot rationally decide which one to choose. On the contrary, it has been shown that if both players are allowed to play entangled quantum strategies, the game admits again three possible Nash Equilibria, but a unique solution. In fact, in this situation both players get the same expected payoff and the solution is represented by the Nash strategy which gives more reward. This new feature is originated from the fact that both players are forced to act in the same way, this being due to the strong correlation present in the entangled form of the solution strategy $\frac{1}{\sqrt{2}}(|OO\rangle + |TT\rangle)$. The meaning of this result is in some sense obvious. If Alice and Bob are obliged to play the same strategy and have the possibility of choosing together sometimes O and sometimes T (this peculiarity not being present in the classical game and in the factorizable quantum case), it is a priori clear that they obtain the best reward playing one-half times O and one-half T. In any case, what is important is the fact that the entangled strategies, in the context of a quantum approach to the Theory of Games, give such a result.

Concluding, we can observe that it could be interesting to apply the quantum version to the study of the dynamic games. In such a case the transformations applied to the strategies represent the moves played sequentially by the players. New results could then come out from some clever choice of these transformations.

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